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# Deformation of Batalin-Vilkovisky Structures

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A Batalin-Vilkovisky formalism is most general framework to construct consistent quantum field theories. Its mathematical structure is called a *Batalin-Vilkovisky structure*. First we explain rather mathematical setting of a Batalin-Vilkovisky formalism. Next, we consider deformation theory of a Batalin-Vilkovisky structure. Especially, we consider deformation of topological sigma models in any dimension, which is closely related to deformation theories in mathematics, including deformation from commutative geometry to noncommutative geometry. We obtain a series of new nontrivial topological sigma models and we find these models have the Batalin-Vilkovisky structures based on a series of new algebroids.

## 1 Introduction

Topological field theory is a powerful method to analyze geometry by means of a quantum field theory.[BBRT] A Batalin-Vilkovisky formalism [BV] is a most general systematic method to treat a consistent quantum field theory. So it is natural to treat topological field theory by means of a Batalin-Vilkovisky formalism.

Deformation theory is one of main topics in mathematics. On the other hand, deformation theory of a quantum field theory is proposed by [BBH][BH] in the physical context. Purposes of this deformation theory are to find a new gauge theory, or to prove a no-go theorem to construct a new gauge theory. Now we apply deformation theory to topological field theories, especially to topological sigma models. Our purpose is to construct many geometries by topological field theories, classify geometries as topological field theories, and unify many deformation theories as a topological field theories. Moreover we can analyze many deformation theories in mathematics as quantum field theories.

In this article, we consider a *topological sigma model*. Let  $X$  and  $M$  be two manifolds. We denote  $\phi$  a (smooth) map from  $X$  to  $M$ . A *sigma model* is a

quantum field theory constructed from a map  $\phi$  (and other auxiliary fields). We analyze structures on  $M$  by the structures induced from  $X$ . A *topological sigma model* is a sigma model independent of metrics of  $X$  and  $M$  [W1].

The AKSZ formulation [AKSZ] of the Batalin-Vilkovisky formalism is a general framework to construct a topological sigma model by the Batalin-Vilkovisky method. This formulation is appropriate to analyze geometry by means of a topological sigma model. In this article, first we generalize the AKSZ formulation to a general  $n$  dimensional base manifold  $X$  and a target manifold with general gradings  $p$ .

Next we discuss deformation of a Batalin-Vilkovisky structure with general gradings. We can construct various geometrical structures on  $M$  if we consider general  $n$  and general grading  $p$ . This formulation provides a clear method to unify and to classify many geometries as Batalin-Vilkovisky structures, and to analyze them as quantum field theories. This construction includes many new topological sigma models as special cases, for examples, the topological sigma model with a Kähler structure ( $A$  model), with a complex structure ( $B$  model) [W2][AKSZ], with a symplectic structure [W1], with a Poisson structure (the Poisson sigma model) [II][SS], with a Courant algebroid structure [R1][I4][HP], with a twisted Poisson structure [KS][I5], with a Dirac structure [KSS], with a generalized complex structure [Z][I5][Pe], and so on.

## 2 AKSZ formulation of Batalin-Vilkovisky formalism on Graded Bundles

We explain general setting of the AKSZ formulation [AKSZ] of the Batalin-Vilkovisky formalism for a general graded bundle in the rather mathematical context.

Let  $M$  be a smooth manifold in  $d$  dimensions. We consider a *supermanifold*  $IIT^*M$ . Mathematically,  $IIT^*M$ , whose bosonic part is  $M$ , is defined as a cotangent bundle with reversed parity of the fiber. That is, a base manifold  $M$  has a Grassman even coordinate and the fiber of  $IIT^*M$  has a Grassman odd coordinate. We can introduce a grading. The coordinate on the base manifold has grade zero and the coordinate on the fiber has grade one. This grading is called the *total degrees*. Similarly, we can define  $IITM$  for a tangent bundle  $TM$ .

We can consider more general assignments for the degree of the fibers of  $T^*M$  or  $TM$ . For a nonnegative integer  $p$ , we define  $T^*[p]M$ , which is called a graded cotangent bundle.  $T^*[p]M$  is a cotangent bundle the degree of whose fiber is  $p$ . If  $p$  is odd, the fiber is Grassman odd, and if  $p$  is even, the fiber is Grassman even. The coordinate on the base manifold has the total degree zero and the coordinate on the fiber has the total degree  $p$ . We define a graded tangent bundle  $T[p]M$  in the same way. For a general vector bundle  $E$ , a

graded vector bundle  $E[p]$  is defined in a similar way.  $E[p]$  is a vector bundle which degree of the fiber is shifted by  $p$ .<sup>1</sup>

We consider a Poisson manifold  $N$  with a Poisson bracket  $\{*, *\}$ . If we can construct a graded manifold  $\tilde{N}$  from  $N$ , then the Poisson structure  $\{*, *\}$  shifts to a graded Poisson structure by grading of  $\tilde{N}$ . The graded Poisson bracket is called an *antibracket* and denoted by  $(*, *)$ .  $(*, *)$  is graded symmetric and satisfies the graded Leibniz rule and the graded Jacobi identity with respect to grading of the manifold. The antibracket  $(*, *)$  with the total degree  $-n+1$  satisfies the following identities:

$$\begin{aligned} (F, G) &= -(-1)^{(|F|+1-n)(|G|+1-n)}(G, F), \\ (F, GH) &= (F, G)H + (-1)^{(|F|+1-n)|G|}G(F, H), \\ (FG, H) &= F(G, H) + (-1)^{|G|(|H|+1-n)}(F, H)G, \\ (-1)^{(|F|+1-n)(|H|+1-n)}(F, (G, H)) &+ \text{cyclic permutations} = 0, \end{aligned} \quad (1)$$

where  $F, G$  and  $H$  are functions on  $\tilde{N}$  and  $|F|, |G|$  and  $|H|$  are the total degrees of functions respectively. The graded Poisson structure is also called the *P-structure*.

Typical examples of a Poisson manifold  $N$  are a cotangent bundle  $T^*M$  and a vector bundle  $E \oplus E^*$ . Two bundles have important roles in this paper. An other example is a vector bundle  $E$  with a Poisson structure on the fiber. We consider these three bundles.

First we consider a cotangent bundle  $T^*M$ . Since  $T^*M$  has a natural symplectic structure, we can define a Poisson bracket induced from the natural symplectic structure. If we take a local coordinate  $\phi^i$  on  $M$  and a local coordinate  $B_i$  of the fiber, we can define a Poisson bracket as follows:<sup>2</sup>

$$\{F, G\} \equiv F \frac{\overleftarrow{\partial}}{\partial \phi^i} \frac{\overrightarrow{\partial}}{\partial B_i} G - F \frac{\overleftarrow{\partial}}{\partial B_i} \frac{\overrightarrow{\partial}}{\partial \phi^i} G. \quad (2)$$

where  $F$  and  $G$  are a function on  $T^*M$ , and  $\overleftarrow{\partial}/\partial \varphi$  and  $\overrightarrow{\partial}/\partial \varphi$  are the right and left differentiations with respect to  $\varphi$ , respectively. Next we shift the degree of fiber by  $p$ , and we consider the space  $T^*[p]M$ . The Poisson structure changes to a graded Poisson structure. The corresponding graded Poisson bracket is called the *antibracket*,  $(*, *)$ . Let  $\phi^i$  be a local coordinate of  $M$  and  $\mathbf{B}_{n-1,i}$  a basis of the fiber of  $T^*[p]M$ .<sup>3</sup> An *antibracket*  $(*, *)$  on a cotangent bundle  $T^*[p]M$  is represented as:

$$(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial \phi^i} \frac{\overrightarrow{\partial}}{\partial \mathbf{B}_{p,i}} G - F \frac{\overleftarrow{\partial}}{\partial \mathbf{B}_{p,i}} \frac{\overrightarrow{\partial}}{\partial \phi^i} G. \quad (3)$$

<sup>1</sup> Note that only the fiber is shifted and the base space is not shifted.

<sup>2</sup> We take Einstein's summation notation.

<sup>3</sup> We use bold notations for local coordinates of graded (super)vector bundles, while we use nonbold notations for local coordinates of usual vector bundles.

The total degree of the  $(*, *)$  is  $-p$ .

Next, we consider a vector bundle  $E \oplus E^*$ . There is a natural Poisson structure on the fiber of  $E \oplus E^*$  induced from natural pairing among  $E$  and  $E^*$ . If we take a local coordinate  $A^a$  on the fiber of  $E$  and  $B_a$  on the fiber of  $E^*$ , we define

$$\{F, G\} \equiv F \frac{\overleftarrow{\partial}}{\partial A^a} \frac{\overrightarrow{\partial}}{\partial B_a} G - F \frac{\overleftarrow{\partial}}{\partial B_a} \frac{\overrightarrow{\partial}}{\partial A^a} G. \quad (4)$$

where  $F$  and  $G$  are a function on  $E \oplus E^*$ . We shift the degrees of fibers of  $E$  and  $E^*$  to  $E[p] \oplus E^*[q]$ , where  $p$  and  $q$  are positive integers. The Poisson structure changes to a graded Poisson structure  $(*, *)$ . Let  $\mathbf{A}_p^a$  be a basis of the fiber of  $E[p]$  and  $\mathbf{B}_{q,a}$  a basis of the fiber of  $E^*[q]$ . An antibracket is represented as

$$(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial \mathbf{A}_p^a} \frac{\overrightarrow{\partial}}{\partial \mathbf{B}_{q,a}} G - (-1)^{pq} F \frac{\overleftarrow{\partial}}{\partial \mathbf{B}_{q,a}} \frac{\overrightarrow{\partial}}{\partial \mathbf{A}_p^a} G. \quad (5)$$

The total degree of the  $(*, *)$  is  $-p - q$ .

Next, we consider a vector bundle  $E$  with a Poisson structure on the fiber. If we shift the degree of the fiber of  $E$  to  $E[p]$ , the Poisson structure changes to a graded Poisson structure  $(*, *)$ . Let  $\mathbf{A}_p^a$  be a basis of the fiber of  $E[p]$ . An antibracket is represented as

$$(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial \mathbf{A}_p^a} k^{ab} \frac{\overrightarrow{\partial}}{\partial \mathbf{A}_p^b} G, \quad (6)$$

where  $F$  and  $G$  are a function on  $E[p]$  and  $k^{ab}$  is a nondegenerate constant bivector induced from a (graded) Poisson structure. The total degree of the antibracket  $(*, *)$  is  $-2p$ .

Next we define a *Q-structure*. A *Q-structure* is a function  $S$  on a supermanifold  $\tilde{N}$  which satisfies the classical master equation  $(S, S) = 0$ .  $S$  is called a *Batalin-Vilkovisky action*, or simply an *action*. We require that  $S$  satisfy the compatibility condition

$$S(F, G) = (SF, G) + (-1)^{|F|+1} (F, SG), \quad (7)$$

where  $F$  and  $G$  are arbitrary functions and  $|F|$  is the total degree of  $F$ .  $(S, F) = \delta F$  generates an infinitesimal transformation (a Hamiltonian flow). We call this a *BRST transformation*, which coincides with the gauge transformation of the theory.

We define a (classical) *Batalin-Vilkovisky structure* as follows:

**Definition 1.** *If a structure on a supermanifold has P-structure and Q-structure, it is called a Batalin-Vilkovisky structure.*

### 3 Batalin-Vilkovisky Structures of Abelian Topological Sigma Models

#### 3.1 BF case

In this section, we consider Batalin-Vilkovisky structures of topological sigma models.

Let  $X$  be a base manifold in  $n$  dimensions, with or without boundary, and  $M$  be a target manifold in  $d$  dimensions. We denote  $\phi$  a smooth map from  $X$  to  $M$ .

We consider a *supermanifold*  $ITX$ , whose bosonic part is  $X$ .  $ITX$  is defined as a tangent bundle with reversed parity of the fiber. We extend a smooth function  $\phi$  to a function  $\phi : ITX \rightarrow M$ .  $\phi$  is an element of  $IT^*X \otimes M$ . The *total degree* defined in the previous section is grading with respect to  $M$ . We introduce a nonnegative integer grading on  $IT^*X$ . A coordinate on a base manifold is zero and a coordinate on the fiber is one. This grading is called the *form degrees*. We denote  $\deg F$  the form degree of a function  $F$ .  $ghF = |F| - \deg F$  is called the *ghost number*.

First we consider a *P-structure* on  $T^*[p]M$ . It is natural to take  $p = n - 1$  to construct a Batalin-Vilkovisky structure in a topological sigma model. In other words, the dimensions of  $X$  labels the total degree of a Batalin-Vilkovisky structure on the supermanifold  $T^*[p]M$ . We consider  $T^*[n - 1]M$  for an  $n$ -dimensional base manifold  $X$ . Let  $\phi^i$  be a local coordinate expression of  $IT^*X \otimes M$ , where  $i, j, k, \dots$  are indices of local coordinate on  $M$ . Let  $B_{n-1,i}$  be a basis of sections of  $IT^*X \otimes \phi^*(T^*[n - 1]M)$ . According to the discussion in the previous section, we can define an *antibracket*  $(*, *)$  on a cotangent bundle  $T^*[n - 1]M$  as

$$(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial \phi^i} \frac{\overrightarrow{\partial}}{\partial B_{n-1,i}} G - F \frac{\overleftarrow{\partial}}{\partial B_{n-1,i}} \frac{\overrightarrow{\partial}}{\partial \phi^i} G, \quad (8)$$

where  $F$  and  $G$  are functions of  $\phi^i$  and  $B_{n-1,i}$ . We take a Darboux coordinate  $\phi^i, B_{n-1,i}$ , but it is for simplicity. We can take more general coordinates. The total degree of the antibracket is  $-n + 1$ . If  $F$  and  $G$  are functionals of  $\phi^i$  and  $B_{n-1,i}$ , we understand an antibracket is defined as

$$(F, G) \equiv \int_X F \frac{\overleftarrow{\partial}}{\partial \phi^i} \frac{\overrightarrow{\partial}}{\partial B_{n-1,i}} G - F \frac{\overleftarrow{\partial}}{\partial B_{n-1,i}} \frac{\overrightarrow{\partial}}{\partial \phi^i} G, \quad (9)$$

where the integration over  $X$  is understood as that on the  $n$ -form part of the integrand. Through this article, we always understand an antibracket on two functionals in a similar manner and abbreviate this notation.

Next we consider a *P-structure* on  $E \oplus E^*$ . Natural assignment of the total degrees is nonnegative integers  $p$  and  $q$  are  $p + q = n - 1$ . That is, we consider  $E[p] \oplus E^*[n - p - 1]$ , where  $1 \leq p \leq n - 2$ . We can naturally construct a

topological sigma model in this case. Let  $\lfloor x \rfloor$  be the floor function which gives the largest integer less than or equal to  $x$ . If  $\lfloor \frac{n}{2} \rfloor \leq p \leq n-2$ , we identify  $E[p] \oplus E^*[n-p-1]$  with the dual bundle  $E^*[n-p-1] \oplus (E^*)^*[p]$ . Therefore  $1 \leq p \leq \lfloor \frac{n-1}{2} \rfloor$  provides different structures of grading.

Let  $\mathbf{A}_p^{a_p}$  be a basis of sections of  $II T^* X \otimes \phi^*(E[p])$  and  $\mathbf{B}_{n-p-1, a_p}$  a basis of the fiber of  $II T^* X \otimes \phi^*(E^*[n-p-1])$ . From (5), we can define an antibracket as

$$(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial \mathbf{A}_p^{a_p}} \frac{\overrightarrow{\partial}}{\partial \mathbf{B}_{n-p-1, a_p}} G - (-1)^{np} F \frac{\overleftarrow{\partial}}{\partial \mathbf{B}_{n-p-1, a_p}} \frac{\overrightarrow{\partial}}{\partial \mathbf{A}_p^{a_p}} G. \quad (10)$$

We want to consider various grading assignments for  $E \oplus E^*$ . Because each assignment induces different Batalin-Vilkovisky structures. In order to consider all independent assignments, we define the following bundle. Let  $E_p$  be  $\lfloor \frac{n-1}{2} \rfloor$  series of vector bundles, where  $1 \leq p \leq \lfloor \frac{n-1}{2} \rfloor$ . We consider  $E_p[p] \oplus E_p^*[n-p-1]$  and consider a direct sum

$$\sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} E_p[p] \oplus E_p^*[n-p-1] \quad (11)$$

And we define a *P-structure* on the graded vector bundle

$$\left( \sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} E_p[p] \oplus E_p^*[n-p-1] \right) \oplus T^*[n-1]M. \quad (12)$$

A local (Darboux) coordinate expression of the antibracket  $(\cdot, \cdot)$  is a sum of (8) and (10):

$$(F, G) \equiv \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} F \frac{\overleftarrow{\partial}}{\partial \mathbf{A}_p^{a_p}} \frac{\overrightarrow{\partial}}{\partial \mathbf{B}_{n-p-1, a_p}} G - (-1)^{np} F \frac{\overleftarrow{\partial}}{\partial \mathbf{B}_{n-p-1, a_p}} \frac{\overrightarrow{\partial}}{\partial \mathbf{A}_p^{a_p}} G \quad (13)$$

where  $p=0$  component is the antibracket (8) on the graded cotangent bundle  $T^*[n-1]M$  and  $\mathbf{A}_0^{a_0} = \phi^i$ . Note that all terms of the antibracket have the total degree  $-n+1$ . We can confirm that the antibracket (13) satisfies the identity (1).

We construct a *Q-structure* on the bundle (12). A simplest and natural action for a topological sigma model is

$$S_0 = \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-p} \int_X \mathbf{B}_{n-p-1, a_p} d\mathbf{A}_p^{a_p}, \quad (14)$$

where  $d$  is a exterior differential on  $X$ . The total degree of  $d$  is 1 because the form degree is 1 and we assign the ghost number 0. The integration over

$X$  is understood as that over the  $n$ -form part (the form degree  $n$  part) of the integrand. This action is analogy of a fundamental form  $\theta = p_i dq^i$  for a symplectic form  $\omega$ , which has  $\omega = -d\theta$ , therefore this action is directly derived from the  $P$ -structure on the graded bundle. The total degree of  $S_0$  is  $n$ . It is called an *abelian BF theory in  $n$  dimensions*.

$S_0$  defines a  $Q$ -structure, since we can easily confirm that  $S_0$  satisfies the classical master equation:

$$(S_0, S_0) = 0, \quad (15)$$

'Abelian' means that the theory has a  $U(1)$  gauge symmetry. The BRST transformation (the gauge symmetry) is defined as

$$\delta_0 \Phi \equiv (S_0, \Phi) = d\Phi, \quad (16)$$

where  $\Phi$  is an arbitrary section of a total bundle  $\left(\sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} E_p[p] \oplus E_p^*[n-p-1]\right) \oplus T^*[n-1]M$ .  $\delta_0^2 = 0$  is satisfied from  $(S_0, S_0) = 0$ , which is consistent to  $d^2 = 0$ .

### 3.2 Chern-Simons with BF case

For a vector bundle  $E$  with a Poisson structure on the fiber, we can construct an another topological sigma model if  $n$  is odd. We consider a graded vector bundle  $E[q]$  for  $E$ , which degree of the fiber is shifted by  $q$ . Let  $\mathbf{A}_q^{a_q}$  be a basis of sections of  $HT^*X \otimes \phi^*(E[q])$ . From the equation (6), we can define an antibracket as

$$(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial \mathbf{A}_q^a} k^{ab} \frac{\overrightarrow{\partial}}{\partial \mathbf{A}_q^b} G, \quad (17)$$

If we take a nonnegative integer  $q = \frac{n-1}{2}$ , then the total degrees of (8), (10) and (17) are all  $-n+1$ . Thus we can consider a combined  $P$ -structure.

We consider a direct sum of  $E[q] = E[\frac{n-1}{2}]$  with  $E_p[p] \oplus E_p^*[n-p-1]$

$$\left( \sum_{p=1}^{\frac{n-3}{2}} E_p[p] \oplus E_p^*[n-p-1] \right) \oplus E\left[\frac{n-1}{2}\right], \quad (18)$$

where we have absorbed  $E_p[p] \oplus E_p^*[n-p-1]$  to  $E[\frac{n-1}{2}]$  for  $p = \frac{n-1}{2}$ . We can define a natural  $P$ -structure on the graded vector bundle

$\left(\sum_{p=1}^{\frac{n-3}{2}} E_p[p] \oplus E_p^*[n-p-1]\right) \oplus E\left[\frac{n-1}{2}\right] \oplus T^*[n-1]M$ . A local (Darboux) coordinate expression of the antibracket  $(\cdot, \cdot)$  is a sum of (8), (10) and (17):

$$\begin{aligned} (F, G) \equiv & \sum_{p=0}^{\frac{n-3}{2}} F \frac{\overleftarrow{\partial}}{\partial \mathbf{A}_p^{a_p}} \frac{\overrightarrow{\partial}}{\partial \mathbf{B}_{n-p-1}^{a_p}} G - (-1)^{np} F \frac{\overleftarrow{\partial}}{\partial \mathbf{B}_{n-p-1}^{a_p}} \frac{\overrightarrow{\partial}}{\partial \mathbf{A}_p^{a_p}} G \\ & + F \frac{\overleftarrow{\partial}}{\partial \mathbf{A}_q^a} k^{ab} \frac{\overrightarrow{\partial}}{\partial \mathbf{A}_q^b} G, \end{aligned} \quad (19)$$

where  $p = 0$  component is the antibracket on the graded cotangent bundle  $T^*[n-1]M$  and  $\mathbf{A}_0^{a_0} = \phi^i$ .  $q = \frac{n-1}{2}$  is needed for all the terms to have the total degree  $-n+1$ .

A simplest and natural action  $S_0$  is

$$S_0 = \sum_{p=0}^{\frac{n-3}{2}} (-1)^{n-p} \int_X \mathbf{B}_{n-p-1 \ a_p} d\mathbf{A}_p^{a_p} + \int_X \frac{k^{ab}}{2} \mathbf{A}_{\frac{n-1}{2} \ a} d\mathbf{A}_{\frac{n-1}{2} \ b}, \quad (20)$$

The second term is called an *abelian Chern-Simons theory* in  $n$  dimensions.  $S_0$  defines a  $Q$ -structure, since we can easily confirm that  $S_0$  satisfies the classical master equation:

$$(S_0, S_0) = 0, \quad (21)$$

This action also has a  $U(1)$  gauge symmetry. The BRST transformation is

$$\delta_0 \Phi = (S_0, \Phi) = d\Phi, \quad (22)$$

where  $\Phi$  is an arbitrary section of the total bundle.

## 4 Deformation

In this section, we consider deformation of Batalin-Vilkovisky structures. Deformation means deformation of the  $Q$ -structure for a fixed  $P$ -structure. We consider local deformation from the settled point (14) or (20) of the moduli space.

First we consider the BF case of the bundle  $\left(\sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} E_p[p] \oplus E_p^*[n-p-1]\right) \oplus T^*[n-1]M$ . The beginning  $Q$ -structure is the equation (14). We deform this Batalin-Vilkovisky action  $S_0$  to

$$S = S_0 + gS_1, \quad (23)$$

under the condition that  $S$  also satisfies the classical master equation:

$$(S, S) = 0, \quad (24)$$

where  $g$  is a deformation parameter and  $S_1$  represents all the deformation terms, which are functional on  $X$  and functions on  $\left(\sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} E_p[p] \oplus E_p^*[n-p-1]\right) \oplus T^*[n-1]M$ . We require that  $S$  is the total degree  $n$ . It is equivalent that the ghost number is zero  $\text{gh}S = |S| - \deg S = 0$ . This condition is physically necessary, though we can relax this condition mathematically. If two deformations  $S$  and  $S'$  satisfy  $S' = S + (S_0, T) = S + \delta_0 T$  for some functional  $T$ , two Batalin-Vilkovisky structures are equivalent. Therefore the problem is to



look for the total degree  $n$  cohomology class

$$H^n \left( \mathcal{H}T^*X \otimes \left( \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} E_p[p] \oplus E_p^*[n-p-1] \right) \oplus T^*[n-1]M \right).$$

Substituting  $S = S_0 + gS_1$  to (24), we obtain  $g$  expansion

$$(S_0, S_0) + 2g(S_0, S_1) + g^2(S_1, S_1) = 0, \quad (25)$$

Zero-th order of the equation (25),  $(S_0, S_0) = 0$ , is already satisfied.

First order is  $(S_0, S_1) = 0$ . Because of the equation (16),  $S_1$  is the integration of an arbitrary function  $F$  of all fundamental superfields  $\Phi$  which are  $A_p^{a_p}$  or  $B_{q_b q}$ , and their derivatives  $d\Phi$ :

$$S_1 = \int_X F(\Phi, d\Phi). \quad (26)$$

For simplicity, we assume that there is no boundary contribution to  $S$ , that is, integration of total derivative terms on  $X$  is always zero. This corresponds to assume that there is no obstruction of deformation. Then we can prove the following theorem:

**Theorem 1.** *Assume there is no boundary contribution on  $X$ , i.e.  $\int_X dG(\Phi) = 0$  for any function  $G$ . If a monomial of  $F(\Phi, d\Phi)$  includes at least one  $d\Phi$ ,  $\int_X F(\Phi, d\Phi)$  is  $\delta_0$ -exact.*

*Proof.* We can assume that  $\int_X F(\Phi, d\Phi) = \sum_{p=0}^{n-1} \int_X F_{n-p-1} dG_p$ , where  $F_{n-p-1}$  are functions with the form degree  $n-p-1$  and  $G_p$  are functions with the form degree  $p$ . From (16), we obtain

$$\begin{aligned} \delta_0 F_0 &= 0, \\ \delta_0 F_{n-p-1} &= dF_{n-p-2} \quad \text{for } -1 \leq p \leq n-2, \\ dF_n &= 0, \\ \delta_0 G_0 &= 0, \\ \delta_0 G_p &= dG_{p-1} \quad \text{for } 1 \leq p \leq n, \\ dG_n &= 0, \end{aligned} \quad (27)$$

For even  $p$ , adjoining two terms are combined as

$$\begin{aligned} &F_{n-p-1} dG_p + F_{n-p-2} dG_{p+1} \\ &= (-1)^{n-p-1} \delta_0(F_{n-p-1} G_{p+1}) - (-1)^{n-p-1} d(F_{n-p-2} G_{p+1}) \end{aligned} \quad (28)$$

using the relations (27). Thus integration of these two terms is  $\delta_0$ -exact.

If  $n$  is even,  $S_1 = \sum_{p=0}^{n-1} \int_X F_{n-p-1} dG_p$  has even numbers of terms, therefore we combine each two term like (28) and we can confirm that  $S_1$  is  $\delta_0$ -exact.

If  $n$  is odd, the last term  $F_0 dG_{n-1}$  remains. However this term is  $\delta_0$ -exact because  $F_0 dG_{n-1} = \delta_0(F_0 G_n)$ . Therefore  $S_1$  is  $\delta_0$ -exact.  $\square$

Therefore since the nontrivial  $S_1$  cohomological class does not include  $d$ , we can take  $S_1 = \int_X F(\Phi)$ . Concretely we can express

$$S_1 = \sum_{p(1), \dots, p(k), q(1), \dots, q(l)} \int_X F_{p(1) \dots p(k), q(1) \dots q(l)}^{a_{p(1)} \dots a_{p(k)} b_{q(1)} \dots b_{q(l)}} (\mathbf{A}_0^{a_0}) \\ \times \mathbf{A}_{p_1}^{a_{p(1)}} \dots \mathbf{A}_{p_k}^{a_{p(k)}} \mathbf{B}_{q_1 b_{q(1)}} \dots \mathbf{B}_{q_l b_{q(l)}}, \quad (29)$$

where  $F_{p(1) \dots p(k), q(1) \dots q(l)}^{a_{p(1)} \dots a_{p(k)} b_{q(1)} \dots b_{q(l)}} (\mathbf{A}_0^{a_0})$  is a function of  $\mathbf{A}_0^{a_0}$  and  $p(r) \neq 0, q(s) \neq 0$  for  $r = 1, \dots, k, s = 1, \dots, l$ . From the  $\deg S = n$  condition, we obtain  $\sum_{\alpha=0}^k |\mathbf{A}_{p_\alpha}^{a_{p(\alpha)}}| + \sum_{\beta=0}^l |\mathbf{B}_{q_\beta b_{q(\beta)}}| = n$ .

Second order of the equation (25)

$$(S_1, S_1) = 0, \quad (30)$$

imposes conditions on functions  $F_{p(1) \dots p(k), q(1) \dots q(l)}^{a_{p(1)} \dots a_{p(k)} b_{q(1)} \dots b_{q(l)}} (\mathbf{A}_0^{a_0})$ . These conditions determine the mathematical structure of a Batalin-Vilkovisky structure.

We call a resulting field theory  $S = S_0 + gS_1$  a *nonlinear gauge theory* in  $n$  dimensions.

Next we consider the Chern-Simons with BF case  $\left( \sum_{p=1}^{\frac{n-3}{2}} E_p[p] \oplus E_p^*[n-p-1] \right) \oplus E\left[\frac{n-1}{2}\right]$ . We make a similar discussion with the BF case. We consider deformation of the Batalin-Vilkovisky action (20) to (23),  $S = S_0 + gS_1$ , under the condition that  $S$  also satisfies the classical master equation (24). We obtain  $S_1$  by using Theorem 1.  $S_1$  has a similar expression with (29) but has different field contents. The second order of the equation (25)

$$(S_1, S_1) = 0, \quad (31)$$

imposes conditions on functions  $F_{p(1) \dots p(k), q(1) \dots q(l)}^{a_{p(1)} \dots a_{p(k)} b_{q(1)} \dots b_{q(l)}} (\mathbf{A}_0^{a_0})$ .

## 5 Deformation in lower dimensions

In this section, we concretely analyze the algebraic and geometric structure of deformation of a topological sigma model in lower dimensional  $X$ .

We can easily find that we cannot obtain nontrivial deformation in case of the total degree  $n = 1$ , thus we cannot obtain a nontrivial structure.

### 5.1 $n = 2$

We analyze the algebraic structure of the total degree  $n = 2$  topological sigma model. In two dimensions, the total graded bundle (12) is  $T^*[1]M = \Pi T^*M$ .

(23) under (29) is

$$S = S_0 + gS_1, \\ S_0 = \int_X \mathbf{B}_{1i} d\phi^i, \quad S_1 = \int_\Sigma \frac{1}{2} f^{ij}(\phi) \mathbf{B}_{1i} \mathbf{B}_{1j}, \quad (32)$$

where  $i, j, \dots$  are indices of a local coordinate expressions on  $T^*[1]M$ . and we rewrite the notations as  $\phi^i = \mathbf{A}_0^i$  and  $\frac{1}{2} f^{ij}(\phi) = F_{,11}{}^{ij}(\mathbf{A}_0)$ . This topological sigma model is known as *the Poisson sigma model* [II][SS]. If we substitute this  $S_1$  to the condition (30), we obtain the geometric structure of the action. We obtain the following identity on  $f^{ij}$ :

$$f^{kl} \frac{\overrightarrow{\partial}}{\partial \phi^l} f^{ij} + f^{il} \frac{\overrightarrow{\partial}}{\partial \phi^l} f^{jk} + f^{jl} \frac{\overrightarrow{\partial}}{\partial \phi^l} f^{ki} = 0. \quad (33)$$

If we restrict the identity on  $X$  (i.e. a ghost number zero sector), (33) reduces to

$$f^{kl}(\phi) \frac{\partial f^{ij}(\phi)}{\partial \phi^l} + f^{il}(\phi) \frac{\partial f^{jk}(\phi)}{\partial \phi^l} + f^{jl}(\phi) \frac{\partial f^{ki}(\phi)}{\partial \phi^l} = 0. \quad (34)$$

Under the identity (33),  $-f^{ij}$  defines a Poisson structure as

$$\{F(\phi), G(\phi)\} \equiv -f^{ij}(\phi) \frac{\partial F}{\partial \phi^i} \frac{\partial G}{\partial \phi^j}, \quad (35)$$

on the space  $M$ . Conversely, if we consider the Poisson structure  $-f^{ij}$  on  $M$ , which satisfies the identity (33), we can define the action (32) consistently. This Poisson bracket (35) is directly constructed if we restrict the derived bracket [Kos] of a Batalin-Vilkovisky structure:

$$\{F(\phi), G(\phi)\} = -F \frac{\overleftarrow{\partial}}{\partial \phi^i} f^{ij}(\phi) \frac{\overrightarrow{\partial}}{\partial \phi^j} G = ((S, F), G), \quad (36)$$

to  $M$  and  $X$ , i.e. to the ghost number zero sector.

The Batalin-Vilkovisky structure of this theory has a structure of the *Lie algebroid*  $T^*[1]\mathcal{M}$  [LO][O], where  $\mathcal{M}$  is a space of a (smooth) map from  $HTX$  to a target space  $M$ .

A *Lie algebroid* is a generalization of a bundle of a Lie algebra over a base manifold  $\mathcal{M}$ . A Lie algebroid over a manifold is a vector bundle  $\mathcal{E} \rightarrow \mathcal{M}$  with a Lie algebra structure on the space of the sections  $\Gamma(\mathcal{E})$  defined by the Lie bracket  $[e_1, e_2]$ ,  $e_1, e_2 \in \Gamma(\mathcal{E})$  and a bundle map (the anchor)  $\rho : \mathcal{E} \rightarrow T\mathcal{M}$  satisfying the following properties:

$$\begin{aligned} 1, & \quad \text{For any } e_1, e_2 \in \Gamma(\mathcal{E}), \quad [\rho(e_1), \rho(e_2)] = \rho([e_1, e_2]), \\ 2, & \quad \text{For any } e_1, e_2 \in \Gamma(\mathcal{E}), F \in C^\infty(\mathcal{M}), \\ & \quad [e_1, Fe_2] = F[e_1, e_2] + (\rho(e_1)F)e_2, \end{aligned} \quad (37)$$

We can derive the bracket of a Lie algebroid from the antibracket and the Batalin-Vilkovisky structure of the  $n = 2$  theory. In our case, a vector bundle

$\mathcal{E}$  is a cotangent bundle  $T^*[1]\mathcal{M}$ . The Lie bracket of the two sections  $e_1$  and  $e_2$  is defined as a derived bracket of the antibracket by

$$[e_1, e_2] \equiv ((S, e_1), e_2), \quad (38)$$

and the anchor is defined by

$$\rho(e)F(\phi) \equiv (e, (S, F(\phi))). \quad (39)$$

Then we can confirm  $[e_1, e_2] = -[e_2, e_1]$ , from  $(e_1, e_2) = 0$  and the graded Jacobi identity of the antibracket. Similarly, a Lie algebroid conditions 1 and 2 on the bracket  $[\cdot, \cdot]$  and the anchor map  $\rho$  are obtained from the classical master equation  $(S, S) = 0$ . From this derived bracket, we directly obtain the following “noncommutative” relation on the coordinates:

$$[\phi^i, \phi^j] = -f^{ij}(\phi). \quad (40)$$

The anchor is a differentiation on functions of  $\phi$  as

$$\rho(\phi^i)F(\phi) = -f^{ij}(\phi) \frac{\overrightarrow{\partial}}{\partial \phi^j} F(\phi). \quad (41)$$

## 5.2 $n = 3$

### BF case

We analyze a nonlinear gauge theory in three dimensions. In this case, the theory defines the topological open 2-brane as a sigma model [I2][Pa].

We consider a supermanifold  $ITT^*X$  whose ghost number zero part is a three-dimensional manifold  $X$ . A base space  $\mathcal{M}$  is the space of a (smooth) map  $\phi$  from  $ITT^*X$  to a target space  $M$ . In  $n = 3$ , the total bundle (11) is a vector bundle  $E[1] \oplus E^*[1]$ . From (12), we consider  $(E[1] \oplus E^*[1]) \oplus T^*[2]M$ . We obtain an antibracket  $(\cdot, \cdot)$  on the space of sections of  $ITT^*X \otimes \phi^*(E[1] \oplus E^*[1] \oplus T^*[2]M)$  if we set  $n = 3$  in (13).

In order to write down the Batalin-Vilkovisky action  $S$ , we take a local basis on  $ITT^*X \otimes \phi^*(E[1] \oplus E^*[1])$  as  $\mathbf{A}_1^a, \mathbf{B}_{1a}$ , which are Darboux coordinates such that  $(\mathbf{A}_1^a, \mathbf{A}_1^b) = (\mathbf{B}_{1a}, \mathbf{B}_{1b}) = 0$  and  $(\mathbf{A}_1^a, \mathbf{B}_{1b}) = \delta^a_b$ . Moreover we introduce  $\mathbf{B}_{2i}$  a section of  $ITT^*X \oplus \phi^*(T^*[2]M)$ .

The total action (23) under (29) is

$$\begin{aligned} S &= S_0 + gS_1, \\ S_0 &= \int_X [-\mathbf{B}_{2i} d\phi^i + \mathbf{B}_{1a} d\mathbf{A}_1^a], \\ S_1 &= \int_X [f_{1a}{}^i(\phi) \mathbf{A}_1^a \mathbf{B}_{2i} + f_2^{ib}(\phi) \mathbf{B}_{2i} \mathbf{B}_{1b} + \frac{1}{3!} f_{3abc}(\phi) \mathbf{A}_1^a \mathbf{A}_1^b \mathbf{A}_1^c \\ &\quad + \frac{1}{2} f_{4ab}{}^c(\phi) \mathbf{A}_1^a \mathbf{A}_1^b \mathbf{B}_{1c} + \frac{1}{2} f_{5a}{}^{bc}(\phi) \mathbf{A}_1^a \mathbf{B}_{1b} \mathbf{B}_{1c} \\ &\quad + \frac{1}{3!} f_6^{abc}(\phi) \mathbf{B}_{1a} \mathbf{B}_{1b} \mathbf{B}_{1c}], \end{aligned} \quad (42)$$

where we set  $f_{1a}{}^i = F_{1,2a}{}^i$ ,  $f_2^{ib} = F_{21}{}^{ib}$ ,  $\frac{1}{3!}f_{3abc} = F_{111,abc}$ ,  $\frac{1}{2}f_{4ab}{}^c = F_{11,1ab}{}^c$ ,  $\frac{1}{2}f_{5a}{}^{bc} = F_{1,11a}{}^{bc}$ ,  $\frac{1}{3!}f_6^{abc} = F_{111}{}^{abc}$ , for clarity. The condition of the classical master equation (30) imposes the following identities on six  $f_i$ 's,  $i = 1, \dots, 6$  [I2]:

$$\begin{aligned}
& f_{1e}{}^i f_2^{je} + f_2^{ie} f_{1e}{}^j = 0, \\
& - \left( \frac{\vec{\partial}}{\partial \phi^j} f_{1c}{}^i \right) f_{1b}{}^j + \left( \frac{\vec{\partial}}{\partial \phi^j} f_{1b}{}^i \right) f_{1c}{}^j + f_{1e}{}^i f_{4bc}{}^e + f_2^{ie} f_{3ebc} = 0, \\
& f_{1b}{}^j \left( \frac{\vec{\partial}}{\partial \phi^j} f_2^{ic} \right) - f_2^{jc} \left( \frac{\vec{\partial}}{\partial \phi^j} f_{1b}{}^i \right) + f_{1e}{}^i f_{5b}{}^{ec} - f_2^{ie} f_{4eb}{}^c = 0, \\
& - f_2^{jb} \left( \frac{\vec{\partial}}{\partial \phi^j} f_2^{ic} \right) + f_2^{jc} \left( \frac{\vec{\partial}}{\partial \phi^j} f_2^{ib} \right) + f_{1e}{}^i f_6^{ebc} + f_2^{ie} f_{5e}{}^{bc} = 0, \\
& - f_{1[a}{}^j \left( \frac{\vec{\partial}}{\partial \phi^j} f_{4bc]}{}^d \right) + f_2^{jd} \left( \frac{\vec{\partial}}{\partial \phi^j} f_{3abc} \right) + f_{4e[a}{}^d f_{4bc]}{}^e + f_{3e[ab} f_{5c]}{}^{de} = 0, \\
& - f_{1[a}{}^j \left( \frac{\vec{\partial}}{\partial \phi^j} f_{5b]}{}^{cd} \right) - f_2^{j[c} \left( \frac{\vec{\partial}}{\partial \phi^j} f_{4ab}{}^{d]} \right) \\
& + f_{3eab} f_6^{ecd} + f_{4e[a}{}^{[d} f_{5b]}{}^{c]e} + f_{4ab}{}^e f_{5e}{}^{cd} = 0, \\
& - f_{1a}{}^j \left( \frac{\vec{\partial}}{\partial \phi^j} f_6^{bcd} \right) + f_2^{j[b} \left( \frac{\vec{\partial}}{\partial \phi^j} f_{5a}{}^{cd]} \right) + f_{4ea}{}^{[b} f_6^{cd]e} + f_{5e}{}^{[bc} f_{5a}{}^{d]e} = 0, \\
& - f_2^{j[a} \left( \frac{\vec{\partial}}{\partial \phi^j} f_6^{bcd]} \right) + f_6^{e[ab} f_{5e}{}^{cd]} = 0, \\
& - f_{1[a}{}^j \left( \frac{\vec{\partial}}{\partial \phi^j} f_{3bcd]} \right) + f_{4[ab}{}^e f_{3cd]}{}_e = 0, \tag{43}
\end{aligned}$$

where  $[\dots]$  on the indices represents the antisymmetrization of them, e.g.,  $\Phi_{[ab]} = \Phi_{ab} - \Phi_{ba}$ .

If we restrict fields to  $X$ , (43) reduces to the following identities:

$$\begin{aligned}
& f_{1e}{}^i f_2^{je} + f_2^{ie} f_{1e}{}^j = 0, \\
& - \frac{\partial f_{1c}{}^i}{\partial \phi^j} f_{1b}{}^j + \frac{\partial f_{1b}{}^i}{\partial \phi^j} f_{1c}{}^j + f_{1e}{}^i f_{4bc}{}^e + f_2^{ie} f_{3ebc} = 0, \\
& f_{1b}{}^j \frac{\partial f_2^{ic}}{\partial \phi^j} - f_2^{jc} \frac{\partial f_{1b}{}^i}{\partial \phi^j} + f_{1e}{}^i f_{5b}{}^{ec} - f_2^{ie} f_{4eb}{}^c = 0, \\
& - f_2^{jb} \frac{\partial f_2^{ic}}{\partial \phi^j} + f_2^{jc} \frac{\partial f_2^{ib}}{\partial \phi^j} + f_{1e}{}^i f_6^{ebc} + f_2^{ie} f_{5e}{}^{bc} = 0, \\
& - f_{1[a}{}^j \frac{\partial f_{4bc]}{}^d}{\partial \phi^j} + f_2^{jd} \frac{\partial f_{3abc}}{\partial \phi^j} + f_{4e[a}{}^d f_{4bc]}{}^e + f_{3e[ab} f_{5c]}{}^{de} = 0,
\end{aligned}$$

$$\begin{aligned}
& -f_{1[a}{}^j \frac{\partial f_{5b]}^{cd}}{\partial \phi^j} - f_2{}^{j[c} \frac{\partial f_{4ab}{}^{d]} }{\partial \phi^j} + f_{3eab} f_6{}^{ecd} + f_{4e[a}{}^{[d} f_{5b]}{}^{c]e} + f_{4ab}{}^e f_{5e}{}^{cd} = 0, \\
& -f_{1a}{}^j \frac{\partial f_6{}^{bcd}}{\partial \phi^j} + f_2{}^{j[b} \frac{\partial f_{5a}{}^{cd]} }{\partial \phi^j} + f_{4ea}{}^{[b} f_6{}^{cd]e} + f_{5e}{}^{[bc} f_{5a}{}^{d]e} = 0, \\
& -f_2{}^{j[a} \frac{\partial f_6{}^{bcd]} }{\partial \phi^j} + f_6{}^{e[ab} f_{5e}{}^{cd]} = 0, \\
& -f_{1[a}{}^j \frac{\partial f_{3bcd]} }{\partial \phi^j} + f_{4[ab}{}^e f_{3cd]e} = 0.
\end{aligned} \tag{44}$$

The algebraic structure (43) (or (44)) is a *Courant algebroid*. A *Courant algebroid* is introduced by Courant in order to analyze the Dirac structure as a generalization of a Lie algebra of vector fields on a vector bundle [C][LWX]. The Batalin-Vilkovisky structure on a Courant algebroid is first analyzed in [R1].

A *Courant algebroid* is a vector bundle  $\mathcal{E} \rightarrow \mathcal{M}$  and has a nondegenerate symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on the bundle, a bilinear operation  $\circ$  on  $\Gamma(\mathcal{E})$  (the space of sections on  $\mathcal{E}$ ), and a bundle map (called the anchor)  $\rho : \mathcal{E} \rightarrow T\mathcal{M}$  satisfying the following properties:

$$\begin{aligned}
1, \quad & e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3), \\
2, \quad & \rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)], \\
3, \quad & e_1 \circ F e_2 = F(e_1 \circ e_2) + (\rho(e_1)F)e_2, \\
4, \quad & e_1 \circ e_2 = \frac{1}{2} \mathcal{D} \langle e_1, e_2 \rangle, \\
5, \quad & \rho(e_1) \langle e_2, e_3 \rangle = \langle e_1 \circ e_2, e_3 \rangle + \langle e_2, e_1 \circ e_3 \rangle,
\end{aligned} \tag{45}$$

where  $e_1, e_2$  and  $e_3$  are sections of  $\mathcal{E}$ , and  $F$  is a function on  $\mathcal{M}$ ;  $\mathcal{D}$  is a map from functions on  $\mathcal{M}$  to  $\Gamma(\mathcal{E})$  and is defined by  $\langle \mathcal{D}F, e \rangle = \rho(e)F$ .

In our nonlinear gauge theory,  $\mathcal{M}$  is the space of a map  $\phi$  from  $\Pi T X$  to  $M$  and  $\mathcal{E}$  is the space of sections of  $\Pi T^* X \oplus \phi^*(E[1] \oplus E^*[1])$ . Let  $e^a$  be a local basis of sections of  $\mathcal{E}$ . We take  $e^a = \mathbf{A}_1{}^a$  or  $\mathbf{B}_1{}_a$ , which are a  $E[1]$  component or a  $E^*[1]$  component respectively. We define a graded symmetric bilinear form  $\langle \cdot, \cdot \rangle$ , a bilinear operation  $\circ$ , a bundle map  $\rho$  and  $\mathcal{D}$  from the antibracket as follows:

$$\begin{aligned}
e^a \circ e^b & \equiv ((S, e^a), e^b), \\
\langle e^a, e^b \rangle & \equiv (e^a, e^b), \\
\rho(e^a)F(\phi) & \equiv (e^a, (S, F(\phi))), \\
\mathcal{D}(\ast) & \equiv (S, \ast).
\end{aligned} \tag{46}$$

Then we can confirm that the classical master equation  $(S, S) = 0$ , which derive the identity (43) on structure functions  $f$ 's, is equivalent to the conditions 1 to 5 of the equation (45). [R1][I4] We calculate the operations  $\circ$  and  $\rho$  on the basis as follows:

$$\begin{aligned}
\mathbf{A}_1^a \circ \mathbf{A}_1^b &= -f_{5c}^{ab}(\phi) \mathbf{A}_1^c - f_6^{abc}(\phi) \mathbf{B}_{1c}, \\
\mathbf{A}_1^a \circ \mathbf{B}_{1b} &= -f_{4bc}^a(\phi) \mathbf{A}_1^c + f_{5b}^{ac}(\phi) \mathbf{B}_{1c}, \\
\mathbf{B}_{1a} \circ \mathbf{B}_{1b} &= -f_{3abc}(\phi) \mathbf{A}_1^c - f_{4ab}^c(\phi) \mathbf{B}_{1c}, \\
\rho(\mathbf{A}_1^a) \phi^i &= -f_2^{ia}(\phi), \\
\rho(\mathbf{B}_{1a}) \phi^i &= -f_{1a}^i(\phi).
\end{aligned} \tag{47}$$

This topological sigma model defines a Courant algebroid structure on the space  $E[1] \oplus E^*[1]$ . We call this model as *the Courant sigma model*.

### Chern-Simons with BF case

Since we can consider Chern-Simons with BF case if  $n$  is odd, In three dimension, we can construct another model. Let  $E$  be a vector bundle with a Poisson structure on the fiber. If we take  $n = 3$  in the equation (18), we obtain  $E[1]$ . A  $P$ -structure is defined on the graded vector bundle  $E[1] \oplus T^*[2]M$  by setting  $n = 3$  in the equation (19). The abelian action is the  $n = 3$  case in the action (20):

$$S_0 = \int_X -\mathbf{B}_{2i} d\phi^i + \frac{k^{ab}}{2} \mathbf{A}_{1a} d\mathbf{A}_{1b}, \tag{48}$$

and deformation is obtained as follows:

$$\begin{aligned}
S &= S_0 + gS_1, \\
S_1 &= \int_X \left( f_{1a}^i(\phi) \mathbf{A}^a \mathbf{B}_i + \frac{1}{6} f_{2abc}(\phi) \mathbf{A}^a \mathbf{A}^b \mathbf{A}^c \right),
\end{aligned} \tag{49}$$

where we rewrite two structure functions  $f_{1a}^i = gF_{11,a}^i$  and  $\frac{1}{6}f_{2abc} = gF_{30,abc}$  for clarity. If we substitute (49) to the condition (30), we obtain the identities on the structure functions  $f_{1a}^i$  and  $f_{2abc}$  as

$$\begin{aligned}
&k^{ab} f_{1a}^i f_{1b}^j = 0, \\
&\left( \frac{\overrightarrow{\partial}}{\partial \phi^j} f_{1b}^i \right) f_{1c}^j - \left( \frac{\overrightarrow{\partial}}{\partial \phi^j} f_{1c}^i \right) f_{1b}^j + k^{ef} f_{1e}^i f_{2fbc} = 0, \\
&\left( f_{1d}^j \frac{\overrightarrow{\partial}}{\partial \phi^j} f_{2abc} - f_{1c}^j \frac{\overrightarrow{\partial}}{\partial \phi^j} f_{2dab} + f_{1b}^j \frac{\overrightarrow{\partial}}{\partial \phi^j} f_{2cda} - f_{1a}^j \frac{\overrightarrow{\partial}}{\partial \phi^j} f_{2bcd} \right) \\
&+ k^{ef} (f_{2eab} f_{2cdf} + f_{2eac} f_{2dbf} + f_{2ead} f_{2bcf}) = 0.
\end{aligned} \tag{50}$$

The identities (50) define a Courant algebroid. In the definition of the Courant algebroid,  $\mathcal{M}$  is the space of a map  $\phi$  from  $ITX$  to  $M$  and  $\mathcal{E}$  is the space of sections of  $IT^*X \oplus \phi^*(E[1])$ . We define a graded symmetric bilinear form  $\langle \cdot, \cdot \rangle$ , a bilinear operation  $\circ$ , a bundle map  $\rho$  and  $\mathcal{D}$  by the antibracket with the same as the equation (46), Then we can confirm that

the classical master equation  $(S, S) = 0$ , which derive the identity (50) on structure functions  $f$ 's, is equivalent to the conditions 1 to 5 of a Courant algebroid (45). [I3] We take the basis of the section of the fiber  $e^a = \mathbf{A}_1^a$ . We calculate the operations  $\circ$  and  $\rho$  on the basis as follows:

$$\begin{aligned}\mathbf{A}^a \circ \mathbf{A}^b &= -k^{ac}k^{bd}f_{2cde}(\phi)\mathbf{A}^e, \\ \langle \mathbf{A}^a, \mathbf{A}^b \rangle &= k^{ab}, \\ \rho(\mathbf{A}^a)\phi^i &= -f_{1c}^i(\phi)k^{ac}.\end{aligned}\tag{51}$$

All deformations of a topological sigma model on the space  $E[1]$  have a Courant algebroid structure.

### 5.3 General $n$

In  $n$  dimensions, the equation (30),  $(S_1, S_1) = 0$ , impose an algebroid structure on the space (11),  $\sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} E_p[p] \oplus E_p^*[n-p-1]$ . The algebroid structure is derived from the Batalin-Vilkovisky structure  $(S_1, S_1) = 0$  of nonlinear gauge theories. Now we obtain an infinite series of algebroids labeled by  $n$ . We call this algebroid an  $n$ -algebroid.

In the previous section, we have found that the  $n = 2$  case defines a Lie algebroid on  $T^*M$  and the  $n = 3$  case defines a Courant algebroid on  $E \oplus E^*$ . For  $n \geq 4$  cases, we can easily calculate algebraic relations but characterization of algebroid structures is still unknown. Higher order generalization has also been discussed in [Se].

In the Chern-Simons case, the equation (30),  $(S_1, S_1) = 0$  impose an algebroid structure on the space (18),  $\left(\sum_{p=1}^{\frac{n-3}{2}} E_p[p] \oplus E_p^*[n-p-1]\right) \oplus E\left[\frac{n-1}{2}\right]$ . In the previous section, we have found that the  $n = 3$  case defines a Courant algebroid on  $E$ .

## 6 Quantum Version of Deformation

In the previous sections, we have considered a classical BV structure. In this section, we discuss quantum version of deformation of a Batalin-Vilkovisky structure. In this section, we discuss the BF case. We can make a similar discussion in the Chern-Simons with BF case.

In order to quantize a gauge theory, we must fix the gauge. Gauge fixing is carried out by adding a gauge fixing term  $S_{GF}$  to the classical action  $S$ . [GPS] The gauge fixed quantum action is  $S_q = S + S_{GF}$ .

We need the *BV Laplacian*. The BV Laplacian is defined as follows:

$$\Delta F \equiv \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\overrightarrow{\partial}}{\partial \mathbf{A}_p^{a_p}} \frac{\overrightarrow{\partial}}{\partial \mathbf{B}_{n-p-1, a_p}} F \tag{52}$$



The BV Laplacian satisfies the following identity:

$$\Delta(F \cdot G) = (\Delta F)G + (-1)^{(n+1)|F|}(F, G) + (-1)^{|F|}F\Delta G, \quad (53)$$

In order for the generating functional to be gauge invariant in the quantum sense, the following quantum master equation is required:

$$(S_q, S_q) - 2i\hbar\Delta S_q = 0, \quad (54)$$

for the quantum action  $S_q$ . In our  $n$ -algebroid topological sigma model, two terms are independently satisfied, i.e.  $\Delta S_q = 0$  and  $(S_q, S_q) = 0$ .

$\mathcal{O}$  is called an observable if an operator  $\mathcal{O}$  satisfies the following equation:

$$(S_q, \mathcal{O}) - i\hbar\Delta\mathcal{O} = 0. \quad (55)$$

Generally, there are two kinds of observables. One is the integration of a local function  $F$  on the boundary  $\partial X$ . Let  $X_r \subset \partial X$  be a  $r$ -cycle on the boundary  $\partial X$ . If  $F$  has the form degree  $r$ , the integration of  $F$  on the  $r$  cycle  $X_r$ :

$$\mathcal{O} = \int_{X_r} F(\Phi) \quad (56)$$

is nontrivial and satisfies (55).

Another observable is constructed from  $\mathbf{A}_0^a$ . We consider a function  $F$  of  $\mathbf{A}_0^a$  and restrict  $F$  on the boundary,  $\mathcal{O}_F \equiv F(\mathbf{A}_0^a)|_{\partial X}$ . We can confirm that the form degree zero part  $\mathcal{O}_F^{(0)}$  of  $\mathcal{O}_F$  is a local observable with ghost number zero on the boundary.

The generating functional is defined by the path integral as

$$Z[\mathcal{O}_k] = \int \prod_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathcal{D}\mathbf{A}_p \mathcal{D}\mathbf{B}_{n-p-1} e^{\frac{i}{\hbar}(S_q + \sum_r J_r \mathcal{O}_r)}, \quad (57)$$

where  $\mathcal{D}\mathbf{A}_p \mathcal{D}\mathbf{B}_{n-p-1}$  is a path integral measure and  $J_k$  are source fields and  $\mathcal{O}_k$  are observables and  $\hbar = g$ .

We consider  $n = 2$  case. Let  $X$  be a two-dimensional disc. Note that classical deformation derives a Poisson structure on  $T^*M$  in  $n = 2$  case. On the other hand, quantum deformation derives the deformation quantization on a Poisson manifold  $M$ . [Kon] The correlation function of two local observables  $\mathcal{O}_f^{(0)}$  and  $\mathcal{O}_g^{(0)}$  derives the Kontsevich's star product formula [CF] on a Poisson manifold:

$$f * g(x) = \int_{\phi(\infty)=x} \mathcal{D}\phi \mathcal{D}\mathbf{B}_1 \mathcal{O}_f^{(0)}(\phi(1)) \mathcal{O}_g^{(0)}(\phi(0)) e^{\frac{i}{\hbar} S_q}, \quad (58)$$

where  $\phi = \mathbf{A}_0$  and  $0, 1, \infty$  are three distinct points at the boundary  $\partial X$ .

If we calculate the same correlation function for  $S_0$ , we obtain the usual product of functions  $f$  and  $g$ :

$$f(x)g(x) = \int_{\phi(\infty)=x} \mathcal{D}\phi \mathcal{D}B_1 \mathcal{O}_f^{(0)}(\phi(1)) \mathcal{O}_g^{(0)}(\phi(0)) e^{\frac{i}{\hbar} S_{0q}}, \quad (59)$$

where  $S_{0q} = S_0 + S_{GF}$ . Therefore quantum deformation in  $n = 2$  is equivalent to the star deformation on  $C^\infty(M)$ .

We can generalize this discussion to higher orders. Deformation  $S_0 \rightarrow S$  derives a generalization of the star deformation to higher dimensions as follows:

$$m_k[\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k] = \int \prod_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathcal{D}A_p \mathcal{D}B_{n-p-1} \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_k e^{\frac{i}{\hbar} S_q}, \quad (60)$$

under the appropriate regularization and the boundary conditions, where  $S$  is the deformation (23) of the abelian topological sigma model and  $\mathcal{O}_r$ 's are two kinds of observables at the boundary. The correlation functions satisfy the Ward-Takahashi identity derived from the gauge symmetry:

$$\int \prod_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathcal{D}A_p \mathcal{D}B_{n-p-1} \Delta \left( \mathcal{O} e^{\frac{i}{\hbar} S_q} \right) = 0, \quad (61)$$

which leads a quantum geometric structure on the space of correlation functions.

## 7 Summary and Outlook

We have discussed deformation of Batalin-Vilkovisky structures of topological sigma models in  $n$  dimensions. We have constructed general theory of most general deformation in general  $n$  dimensions. We have analyzed structures in the case of  $n = 2$  and 3 in detail. In  $n = 2$  deformation of a BV structure produces a Lie algebroid structure, and in  $n = 3$ , deformation produces a Courant algebroid structure. For  $n \geq 4$ , characterization of  $n$ -algebroids obtained by deformation of topological sigma models is still unknown and an open problem.

We have also discussed quantum version of deformation. For  $n = 2$  case, the deformation on the disc  $X$  is equivalent to the deformation quantization on a Poisson manifold  $M$ .

In  $n = 2$ , there are two special important cases of deformations. They are  $A$ -model and  $B$ -model. [W2] There are many investigations to analyze quantum moduli. For reviews, [BCOV], [HKKPTVVZ] and references therein.

$n = 3$  quantum deformation is analyzed in [HM]. For general  $n \geq 4$ , quantum structures are unknown. If we analyze higher  $n$  cases, we will obtain interesting mathematical and physical structures.

In this article, we assume the  $p$  and  $n - p - 1$  are nonnegative integers in  $E[p] \oplus E^*[n - p - 1]$ , where we identify the  $p = 0$  bundle with a cotangent bundle  $T^*[n - 1]M$ . We will be able to generalize our discussions to negative integers  $p$  and  $n - p - 1$ . A special case has been analyzed in [BM][II2].

We need make analysis of all moduli, i.e. we should consider Kodaira-Spencer theory of Batalin-Vilkovisky structures.

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